AdS/CFT duality for non-relativistic field theory

Walter D. Goldberger

Department of Physics, Yale University,
New Haven, CT 06520, U.S.A.

E-mail: walter.goldberger@yale.edu

ABSTRACT: We formulate a correspondence between non-relativistic conformal field theories (NRCFTs) in $d-1$ spatial dimensions and gravitational theories in AdS_{d+2} backgrounds with one compactified lightlike direction. The breaking of the maximal $SO(2, d+1)$ symmetry of AdS_{d+2} to the non-relativistic conformal group arises from boundary conditions on bulk fields, without the need to introduce non-vacuum sources of energy-momentum. As a check of the proposal, we use the gravitational theory to reproduce the NRCFT state-operator correspondence between scaling dimensions of primary operators and energy eigenstates of the non-relativistic system placed in an external harmonic potential.

Contents

1 Introduction

Non-relativistic quantum systems with two-body bound state energies tuned close to threshold seem to describe a variety of physical problems (see [1] for a review). This tuning may be accidental, as it is in the effective theories that describe the few-body strong interactions of nucleons at low momentum transfers, or arranged by experimental manipulation, as with cold atoms placed in an external confining trap [2]. As the binding energy approaches threshold, the scattering length diverges and the precise nature of the two-body interactions becomes irrelevant. In this limit, the dynamics can be formulated in terms of local non-relativistic scale invariant field theories (NRCFTs), in which the usual Galilean invariance of the interactions is enhanced to a non-relativistic conformal symmetry known as Schrodinger symmetry.[1]

Because the scattering length diverges, the NRCFTs relevant to nature are strongly coupled and thus cannot be treated by the usual perturbative techniques. While it has been possible to obtain analytical results in the few-body sector (for a review of methods see [2, 10]), the many-body properties of these theories have not yielded to analytical methods. Instead one must turn to numerical simulations in order to account for phenomena observed in the laboratory.

---

[1]Schrodinger invariant quantum field theories were first discussed in refs. [3, 4]. In $d = 4$ spacetime dimensions, an important example for both nuclear and atomic physics is the theory of fermions at unitarity, consisting of spin$-1/2$ particles interacting through a short range potential. This theory was shown to satisfy the Ward identities of the Schrodinger group in [5], and the renormalization group scaling of theories close to this fixed point was first described in [6]. For lower dimensional examples see ref. [7]. Formal aspects of Schrodinger invariant field theories are discussed in [8, 9]

In the case of strongly coupled, relativistic (super) conformal field theories, it has been possible in some cases to use the AdS/CFT correspondence [11–13] to recast the dynamics in terms of a dual gravitational theory that, in a suitable limit, becomes weakly coupled. It is therefore natural to ask if a similar mapping can be obtained for strongly coupled non-relativistic conformal theories. Such a gravitational description is likely to yield robust results only when both sides of the duality have enough supersymmetry, so it is perhaps not directly applicable to NRCFTs that can be realized in the laboratory. Nevertheless, establishing a weakly coupled dual gravitational description may still be useful. For example, it might shed light on aspects of the dynamics that are universal and thus independent of any underlying supersymmetry.

A first step towards establishing the correspondence between NRCFT_d (with $d-1$ the number of spatial dimensions) and $(d+2)$-dimensional gravity theory was taken recently by Son [14] and by Balasubramanian and McGreevy [15]. In these papers, the usual AdS/CFT dictionary [12, 13] is adapted to derive NRCFT correlators from a gravitational theory propagating in a background with isometry group $Sch(d-1)$, the d-dimensional Schrodinger group. In order to generate this background, it is necessary include an energy-momentum term in the Einstein equations that breaks the maximal symmetry of the vacuum down to $Sch(d-1)$. The resulting metric in [14, 15] describes a static cosmological model driven by a pressureless fluid, and by applying the rules of AdS/CFT to bulk fields in this geometry, it is possible to reproduce the boundary correlators of Schrodinger invariant theories.

Here we propose instead a version of the correspondence in which the gravity side lives in a (locally) AdS_{d+2} background. The breaking of the full $\text{SO}(2, d+1)$ isometry group is achieved by imposing boundary conditions on AdS bulk fields that only preserve $Sch(d-1)$. As in [14, 15], our prescription yields Schrodinger invariant field theory correlators on the boundary.

There may be some advantages to formulating the gravity/NRCFT correspondence directly in AdS, without introducing additional non-vacuum energy-momentum (i.e., besides a pure cosmological constant) to break the full conformal symmetry. For instance, it may be simpler to describe properties of NRCFT whose dual gravity description depend on the global structure of the spacetime (e.g. the state-operator map [13, 16], phase transitions [17]). Some of these global properties will be discussed below. In addition, AdS backgrounds emerge naturally in supergravity. One may speculate that this raises the possibility of constructing explicit supersymmetric examples of AdS/NRCFT by taking the non-relativistic limit, along the lines discussed below, of standard AdS/CFT duality (supersymmetric extensions of the Schrodinger algebra have been studied in [18]).

In section 3.1 we set up the basic dictionary between NRCFT correlators and AdS bulk physics in the Poincare patch. We use the embedding of $Sch(d-1)$ into $\text{SO}(2, d+1)$ discussed in [19, 21, 22] (see also [14]) to identify modes of AdS_{d+2} bulk fields of definite non-zero momentum along a compactified lightlike direction with boundary NRCFT primary operators. This yields a prescription for computing boundary correlators similar to that in [14, 15]. In section 3.2 we discuss global aspects of the correspondence. In particular, we develop the gravitational description of global Hamiltonian time evolution, and use it to reproduce the state-operator correspondence of NRCFTs established in [9] (see also ref. [23]

for a generalization). We also use the global description to recover the bound $\Delta \geq (d-1)/2$ of scaling dimensions in unitary NRCFTs. By considering asymptotically-AdS black holes, it is possible to extend the methods discussed here to the case of finite temperature/density NRCFTs.

2 Non-relativistic CFTs

A non-relativistic CFT is defined as field theory in $d-1$ spatial dimensions that is invariant under the symmetry group $Sch(d-1)$. In this section we review the necessary facts about $Sch(d-1)$ symmetry and its relation to the conformal group $SO(2, d+1)$ that will be needed in the construction of AdS duals later on.

2.1 Schrodinger invariance

The relation between NRCFTs in $d-1$ spatial dimension and AdS_{d+2} spacetime is closely connected to the relation between the Galilean conformal symmetry of $(d-1)$-dimensional space, $Sch(d-1)$ and the conformal group $SO(2, d+1)$ of $(d+1)$-dimensional Minkowski space. This connection is well known [19–22], and it is perhaps best illustrated in the context of a simple example [24, 25]. Consider a massless free scalar propagating in $d+1$-dimensional Minkowski space,

$$S = \frac{1}{2} \int d^{d+1}x \, \partial_\mu \phi \partial^\mu \phi. \tag{2.1}$$

This is trivially invariant under the conformal group $SO(2, d+1)$. Choose lightlike coordinates (t, ξ) in Minkowski space, with

$$t = \frac{x^0 + x^d}{\sqrt{2}},$$
$$\xi = \frac{x^0 - x^d}{\sqrt{2}}. \tag{2.2}$$

The metric takes the form (with the notation $\vec{x} \cdot \vec{y} = x^i y^i$, $i = 1, \ldots, d-1$),

$$ds^2 = -2dt d\xi + d\vec{x} \cdot d\vec{x}. \tag{2.3}$$

Concentrating on a single mode of ϕ with definite momentum in the ξ direction, $\phi = e^{-i\xi m} \psi(t, \vec{x})$, the action reduces to a free non-relativistic field theory (after renormalizing ψ),

$$S \rightarrow \int dt d\vec{x} \, \psi^\dagger(t, \vec{x}) \left(i\partial_t + \frac{1}{2m} \nabla^2 \right) \psi(t, \vec{x}), \tag{2.4}$$

which is invariant under the group of non-relativistic conformal transformations, generated by the usual Galilean generators ($H, P^i, K^i =$ boosts, $M^{ij} =$ rotations), together with a dilatation D, a Galilean special conformal generator C, and a central charge N corresponding to the total particle number. For example, under dilatations, $D : (t, \vec{x}) \rightarrow (\lambda^2 t, \lambda \vec{x})$ for some λ. The complete algebra is given below in eqs. (2.10)–(2.12).

This example shows that $Sch(d-1)$, the symmetry group of a free non-relativistic field theory (or more generally of a non-relativistic CFT), can be viewed as the subgroup of $SO(2, d+1)$ that does not mix modes with different momentum N along the null direction ξ. In other words, it is the subgroup of $SO(2, d+1)$ that leaves a fixed lightlike momentum vector invariant [19].

In this paper, we will take this as the definition of $Sch(d-1)$. Let the invariant lightlike vector be

$$N = \frac{P^0 - P^d}{2}. \tag{2.5}$$

Then the $Sch(d-1)$ algebra consists of $SO(2, d+1)$ generators that commute with N. The algebra of the conformal group is spanned by the Poincare generators $(P^\mu, M^{\mu\nu})$ along with dilatations $\tilde{D}$ and special conformal transformations K^μ. It is given by

$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu),$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} + \text{perms.}, \tag{2.6}$$
$$[\tilde{D}, M_{\mu\nu}] = 0 \qquad\qquad [\tilde{D}, K_\mu] = iK_\mu$$
$$[\tilde{D}, P_\mu] = -iP_\mu, \tag{2.7}$$
$$[P_\mu, K_\nu] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu}\tilde{D}. \tag{2.8}$$

The subalgebra that commutes with N consists of the generators

$$H = \frac{1}{2}(P^0 + P^d),\ C = \tfrac{1}{2}(K^0 - K^d),\ D = M_{0d} + \tilde{D}, \tag{2.9}$$

as well as spatial translations P^i, boosts $K^i = M^{0i} - M^{di}$, and spatial rotations M^{ij}. These generators satisfy the (extended) $Sch(d-1)$ algebra. whose non-zero commutators are

$$[M_{ij}, P_k] = -i(\delta_{ik}P_j - \delta_{jk}P_i), \quad [M_{ij}, K_k] = -i(\delta_{ik}K_j - \delta_{jk}K_i),$$
$$[M_{ij}, M_{rs}] = -i\delta_{ir}M_{js} + \text{perms.}, \tag{2.10}$$
$$[D, P^i] = -iP^i, \qquad\qquad [D, K^i] = iK^i, \qquad\qquad [C, P^i] = iK^i, \tag{2.11}$$
$$[D, H] = -2iH, \qquad\qquad [D, C] = 2iC, \qquad\qquad [H, C] = iD. \tag{2.12}$$

and

$$[P^i, K^j] = -iN\delta^{ij} \tag{2.13}$$

Note in particular that the generators H, C, D form an $SL(2, \mathbf{R})$ subalgebra. The eigenstates of the operator N can be interpreted as states of definite particle number.

2.2 State-operator correspondence

In relativistic CFTs, there is a one-to-one map between local operators $\mathcal{O}(x)$ with definite scaling dimension Δ and the eigenstates of the conformal Hamiltonian $\mathcal{H} = (P^0 + K^0)/2$ [26]. Roughly speaking, given an operator $\mathcal{O}(x)$, the state $|\mathcal{O}\rangle = \lim_{x\to 0} \mathcal{O}(x)|0\rangle$ is an energy eigenstate, $\mathcal{H}|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle$.

Nishida and Son [9] established an analog of the state-operator correspondence for NRCFTs, which we now review. Let $|0\rangle$ be the vacuum state of H and $\mathcal{O}(t,\vec{x})$ an NRCFT primary operator, which by definition satisfies

$$[K_i, \mathcal{O}(0)] = 0, \ [C, \mathcal{O}(0)] = 0, \ [D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0). \tag{2.14}$$

Then it follows from the $Sch(d-1)$ algebra that the state

$$|\mathcal{O}\rangle = e^{-H/\omega}\mathcal{O}^\dagger(0)|0\rangle, \tag{2.15}$$

is an eigenstate of the deformed Hamiltonian $H_o = H + \omega^2 C$, where ω is a constant with units of frequency,

$$H_o|\mathcal{O}\rangle = \omega\Delta|\mathcal{O}\rangle. \tag{2.16}$$

The deformed Hamiltonian has a useful physical interpretation, which can be exhibited by noting that in a NRCFT, the special conformal generator C can be realized in terms of the fundamental fields $\psi(t,\vec{x})$ as the operator

$$C = \frac{1}{2}\int d\vec{x}\ \psi(t,\vec{x})^\dagger\vec{x}^2\psi(t,\vec{x}). \tag{2.17}$$

Thus the deformed Hamiltonian H_o corresponds to placing the original NRCFT, with internal Hamiltonian H, in a harmonic external potential (setting to unity the mass of the particle excitations associated with the field operator $\psi(t,\vec{x})$). The state-operator map is therefore the statement that the eigenstates of the NRCFT in the harmonic trap are in one-to-one correspondence with the primary operators of the conformal theory, where the energy eigenvalues are the scaling dimensions.[2] The SL(2, $\mathbf{R}$) subalgebra of $Sch(d-1)$ implies that above each state $|\mathcal{O}\rangle$, with $\mathcal{O}$ primary, there is a tower of H_o eigenstates with level spacing 2ω. Different states in this tower are connected by the raising/lowering operators

$$\omega L_\pm = H - \omega^2 C \mp i\omega D, \tag{2.18}$$

which satisfy the commutation relations $[H_o, L_\pm] = \pm 2\omega L_\pm$ as a consequence of the $Sch(d-1)$ algebra. In addition, the descendant operators $\mathcal{O}_{i_1\cdots i_\ell}(0) = \partial_{i_1}\cdots\partial_{i_\ell}\mathcal{O}(0) -$ traces generate SL(2, $\mathbf{R}$) towers above a base state with energy $\Delta + \ell$ in units of ω.

The relation between time evolution generated by H_o and the presence of a harmonic potential has a geometric interpretation, as pointed out in [25, 27]. Following that reference, introduce the following coordinates on $(d+1)$-dimensional flat space:

$$\begin{aligned}
\omega t_o &= \tan^{-1}\omega t,\\
\xi_o &= \xi - \frac{1}{2}\omega\vec{x}^2\frac{\omega t}{1+\omega^2 t^2},\\
\vec{x}_o &= \frac{\vec{x}}{\sqrt{1+\omega^2 t^2}}.
\end{aligned} \tag{2.19}$$

---

[2]In addition to the correspondence between eigenstates of H_o and the operator algebra in the free-space theory introduced in [9], there is an additional correspondence between H_o eigenstates and zero-energy homogeneous N-particle eigenfunctions of H [28]. As shown in ref. [23], these, and many other such mappings, are related by inner automorphisms of the SL(2, $\mathbf{R}$) subalgebra of the Schrodinger group.

These new coordinates have simple transformation properties under the action of H_o,

$$H_o : (t_o, \xi_o, \vec{x}_o) \rightarrow (t_o + \tau, \xi_o, \vec{x}_o) \tag{2.20}$$

for some constant τ. In these coordinates, the metric eq. (2.3) becomes $ds^2 = \cos^2 \omega t_o ds_o^2$, with

$$ds_o^2 = -\omega^2 \vec{x}_o^2 dt_o^2 - 2dt_o d\xi_o + d\vec{x}_o^2. \tag{2.21}$$

By covariantizing eq. (2.1), and adding a term proportional to $\int d^{d+1} x \sqrt{g} R \phi^2$ with a suitably chosen coefficient, the theory becomes invariant under arbitrary conformal transformations of $ds^2 \rightarrow \Omega^2 ds^2$ of the background metric. Making a transformation that converts eq. (2.3) to the metric in eq. (2.21), and plugging in the momentum eigenmode $\phi = e^{-im\xi_o} \psi(t_o, \vec{x}_o)$ into the conformally invariant free Lagrangian, one obtains (dropping the subscript o)

$$S \rightarrow \int dt d\vec{x} \, \psi^\dagger(t, \vec{x}) \left(i\partial_t + \frac{1}{2m} \nabla^2 - \frac{1}{2} m \omega^2 \vec{x}^2 \right) \psi(t, \vec{x}), \tag{2.22}$$

which is the free NRCFT coupled to an external harmonic potential.

3 AdS/CFT for non-relativistic field theories

In this section we establish the correspondence between NRCFT$_d$ and gravity in AdS_{d+2}. In order to generate a background with the requisite symmetry, one must break the full $SO(2, d+1)$ isometry group of AdS down to a $Sch(d-1)$ subgroup. This can be accomplished by selecting a preferred lightlike direction in AdS generated by the orbits of the particle number operator N in the algebra of $Sch(d-1) \subset SO(2, d+1)$. The discreteness of the spectrum of N suggests that the preferred lightlike direction should be compact.

The Lagrangian form of the correspondence, suitable for computing correlation functions is developed in below in section 3.1. The procedure is the standard AdS/CFT prescription for Green's functions [12, 13] (also employed in the approach of [14, 15]), with a modification of the asymptotic behavior of bulk fields near the AdS boundary to account for the symmetry breaking $SO(2, d+1) \rightarrow Sch(d-1)$. In section 3.2 we discuss the global Hamiltonian formulation. There we show how the NRCFT analog of radial quantization (i.e., the state-operator map of ref. [9]) emerges in the AdS picture.

3.1 Poincare coordinates: correlation functions

Consider a local NRCFT primary operator $\mathcal{O}_M$ (i.e., satisfying eq. (2.14)) with particle number M and scaling dimension Δ,

$$[N, \mathcal{O}_M] = -iM\mathcal{O}_N, \quad [D, \mathcal{O}_M(0)] = -i\Delta\mathcal{O}_M(0). \tag{3.1}$$

The generating function of $\mathcal{O}_M(t, \vec{x})$ correlators is

$$Z_{NRCFT}[\phi] = \left\langle T \exp \left[i \int dt d\vec{x} \, \phi(t, \vec{x}) \mathcal{O}_M(t, \vec{x}) + \text{h.c.} \right] \right\rangle. \tag{3.2}$$

where the expectation value is taken in the vacuum of the generator H. We would like to establish a dictionary between correlation functions in the NRCFT$_d$ and quantities in AdS_{d+2}.

To set up the correspondence, it is convenient to use coordinate on AdS_{d+2} that point along the orbits of the generators H, corresponding to time evolution in the NRCFT (in the absence of external fields), and particle number N. Because both these operators are linear combinations of translations P^μ, we work in the Poincare coordinates $X = (z, x^\mu)$ on AdS which transform as $P^\mu : (x^\mu, z) \to (x^\mu + a^\mu, z)$. The metric in these coordinates is

$$ds^2 = \frac{\ell^2}{z^2} \left(-2dtd\xi + d\vec{x} \cdot d\vec{x} + dz^2 \right) \tag{3.3}$$

with ℓ the AdS radius. The null coordinates (t, ξ) are given in terms of Poincare coordinates by eq. (2.2), and simply shift under the action of H, N:

$$H : (t, \xi) \to (t + \Delta t, \xi), \ N : (t, \xi) \to (t, \xi + \Delta \xi). \tag{3.4}$$

Because particle number $N = i\partial_\xi$ is discrete, we assume that the null direction parameterized by ξ is compactified on a circle. This is achieved by identifying $\xi \sim \xi + \lambda$ for some constant λ. There is no invariant definition of radius for lightlike compactification, and indeed the compactification scale does not enter any of our results. Note that the metric on the boundary, $z \to 0$, becomes the flat metric eq. (2.3) up to an overall (divergent) factor.

The correspondence for correlation functions follows from the usual AdS/CFT dictionary between operators $\mathcal{O}$ and AdS bulk fields,

$$Z_{NRCFT}[\phi] = Z_{AdS}[\phi]. \tag{3.5}$$

Here the right hand term is the effective action for a bulk field $\phi(X)$ in AdS_{d+2} subject to the following asymptotics on the AdS boundary at $z \to 0$,

$$\phi(X) \to z^\Delta e^{-iM\xi} \phi(t, \vec{x}). \tag{3.6}$$

Assuming that the semiclassical approximation holds, one can replace $\ln Z_{AdS}[\phi] \simeq iS_c[\phi]$, where $S_c[\phi]$ is the classical gravity action in AdS.

To illustrates how this works, consider a free massive scalar field ϕ propagating in AdS_{d+2}. In the above coordinates the action is,

$$S = -\int d^{d+2}X \sqrt{g} \left[z^2 \eta^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi + z^2 |\partial_z \phi|^2 + m^2 |\phi|^2 \right]. \tag{3.7}$$

We set out to find a solution of the equation of motion with the boundary asymptotics eq. (3.6). This must be of the form

$$\phi(X) = e^{-iM\xi} \phi_M(t, \vec{x}, z). \tag{3.8}$$

The equation of motion becomes

$$-\partial_z^2 \phi_M + \frac{d}{z} \partial_z \phi_M - (2iM\partial_t + \nabla^2)\phi_M + \frac{m^2}{z^2}\phi_M = 0. \tag{3.9}$$

Note that near the boundary, the solutions behave as $\phi_M \sim z^{\Delta_\pm}$, with

$$\Delta_\pm = \frac{d+1}{2} \pm \sqrt{\left(\frac{d+1}{2}\right)^2 + m^2},\tag{3.10}$$

which is the standard AdS/CFT relation between bulk mass and scaling dimension Δ. A similar relation between NRCFT scaling dimensions and bulk quantities was given in refs. [14, 15]. However, in our case the quantity $\Delta_\pm$ which governs the asymptotic behavior of the scalar field is independent of the eigenvalue M of momentum in the ξ direction.[3]

Barring this discrepancy in the definition of m^2 appearing in eq. (3.10), the wave equation eq. (3.9) for an eigenstate of ξ momentum is identical to the corresponding field equations in [14, 15]. This must be the case given that their background metric has $Sch(d-1)$ symmetry by construction. In terms of Fourier modes along the $(t, \vec{x})$ directions, the solutions are as in, e.g. [14],

$$\phi_M \sim z^{\frac{d+1}{2}} K_\nu(Qz),\tag{3.12}$$

with $\nu = \pm\sqrt{(d+1)^2/4 + m^2}$, and $Q = (\vec{p}^2 - 2M\omega)^{1/2}$.

Given the $z \to 0$ asymptotics of the Bessel function $K_\nu(z)$, the boundary behavior is indeed $\phi \sim z^{\Delta_\pm}$. As discussed in [29], there is an ambiguity in the choice of root $\Delta_\pm$ for some values of the mass parameter m. For $m^2 > 1 - (d+1)^2/4$, finiteness of the (Euclidean) action fixes the choice Δ_+, while for

$$-\left(\frac{d+1}{2}\right)^2 < m^2 < 1 - \left(\frac{d+1}{2}\right)^2,\tag{3.13}$$

both Δ_+ and Δ_- are permissible. The interpretation is that for each choice of root Δ, the same AdS theory gives rise to a different boundary CFT [29]. This interpretation carries over to NRCFTs, as discussed in [14]. Note in particular that the smallest possible operator dimension is $\Delta = (d-1)/2$, the dimension of a free non-relativistic field. We will give a more careful derivation of this bound, along the lines of [30], in section 3.2 below from the point of view of Hamiltonian evolution.

Correlation functions of the primary operator $\mathcal{O}_M$ dual to ϕ can now be constructed using eq. (3.5) and eq. (3.12). The calculation (see [14, 15]) is standard given the solutions eq. (3.12) and will not be repeated here. For example, the two-point function, up to some normalization constant c, is given by

$$\langle T\mathcal{O}_M(t, \vec{x})\mathcal{O}_M^\dagger(0, 0)\rangle = c\theta(t)t^{-\Delta} \exp\left(iM\frac{\vec{x}^2}{2t}\right),\tag{3.14}$$

---

[3]It is possible to modify eq. (3.10) to include dependence on the momentum eigenvalue M. This would arise, e.g., from couplings to dynamical fields that get expectation values in the ξ direction. Such VEVs would shift the Lagrangian by the additional terms,

$$S_\xi = -\int d^{d+2}X\sqrt{g}\left[c_1\phi^\dagger i\partial_\xi\phi + c_2|\partial_\xi\phi|^2 + \cdots\right],\tag{3.11}$$

which have the effect of replacing $m^2 \to m^2 + c_1 M + c_2 M^2 + \cdots$ in eq. (3.10).

where Δ is one of the roots in eq. (3.10). This is consistent with the results of [8, 9] for the form of the two-point correlator of a primary operator $\mathcal{O}_M$ in an NRCFT. Comparing eq. (3.14) with the propagator of a free particle of mass M in $d-1$ space dimensions

$$\langle \vec{x}|e^{-iHt}|\vec{x}=0\rangle = \left(\frac{M}{2\pi i}\right)^{\frac{d-1}{2}} \theta(t)t^{-\frac{(d-1)}{2}} \exp\left(iM\frac{\vec{x}^2}{2t}\right), \qquad (3.15)$$

suggests that that the M-particle bound state interpolated by $\mathcal{O}_M^\dagger$ can be interpreted as a free particle of mass M moving in a fractional 2Δ-dimensional space.[4] Connected correlators with more operator insertions encode scattering amplitudes. These amplitudes are parameterized by the various coupling constants (e.g, gravitational couplings) in the AdS bulk and thus scale roughly like powers of $G_N\ell^{-d}$, with G_N the $(d+2)$-dimensional Newton constant. If the AdS radius ℓ is large in Planck units, these correlation functions can be computed from the usual AdS/CFT bulk-to-boundary diagrams, with boundary conditions as in eq. (3.6) for each operator insertion.

3.2 State-operator correspondence

In relativistic CFT_{d+1}s, time evolution generated by the conformal Hamiltonian $\mathcal{H}$, whose eigenstates are the primary operators $|\mathcal{O}\rangle$, is equivalent to canonical quantization on the space $\mathbf{R} \times \mathbf{S}^d$ [26]. The scale introduced by the radius of the sphere $\mathbf{S}^d$ has the effect of generating a finite gap in the spectrum of CFT scaling dimensions. $\mathbf{R} \times \mathbf{S}^d$ is also the global boundary of AdS_{d+2}, where $\mathbf{R}$ is spanned by AdS global time τ. Given the correspondence between CFT operators and AdS bulk fields, this implies a map between AdS field eigenmodes of $\mathcal{H} = i\partial_\tau$ and the states $|\mathcal{O}\rangle$ of the boundary CFT [13, 16]. The spectrum of $i\partial_\tau$ is quantized due to the presence of a gravitational potential well for modes in AdS, and therefore matches the discreteness of the CFT spectrum on $\mathbf{R} \times \mathbf{S}^d$.

In this section, we discuss the AdS interpretation of analogous global aspects of NR-CFTs. In particular, we develop the correspondence between the state-operator map introduced in [9] and the eigenmodes of bulk fields with respect to a suitably chosen global time variable. From this analysis, we also recover the unitarity bound $\Delta \geq (d-1)/2$ on NRCFT_d scaling dimensions. The consistency of these results with those of section 3.1 (e.g., the relation eq. (3.10) between scaling dimensions and AdS field masses) provides a check of the AdS/NRCFT dictionary.

3.2.1 Coordinates

We need to introduce a time coordinate on AdS_{d+2} that points along orbits of the oscillator Hamiltonian $H_o = H + \omega^2 C$ discussed in section 2.2. A suitable set of coordinates can be obtained by starting with the definition of AdS_{d+2} as the quadric surface

$$-\ell^2 = -2X^+X^- - 2Z^+Z^- + X^iX^i \quad (i=1,\ldots,d-1) \qquad (3.16)$$

---

[4]Accordingly, the result in eq. (2.16) can be viewed as the ground state energy of a 2Δ-dimensional harmonic oscillator.

embedded in $\mathbf{R}^{d+3}$ with metric

$$ds^2 = -2dX^+dX^- - 2dZ^+dZ^- + dX^idX^i. \tag{3.17}$$

We parameterize a given point in AdS in terms of coordinates (t, ξ) generated by the orbits of H_o and N respectively and a set of spatial coordinates. A given point in AdS_{d+2} can be written as $(0 \leq \omega t < 2\pi, -\infty < \xi < \infty, x^\pm > 0)$

$$
\begin{pmatrix}
X^i \\
\omega Z^- \\
X^+ \\
Z^+ \\
\omega X^-
\end{pmatrix}
= e^{-itH_o} e^{-i\xi N}
\begin{pmatrix}
x^i \\
0 \\
x^+ \\
0 \\
\omega x^-
\end{pmatrix},
\tag{3.18}
$$

where $x^i, x^\pm$ satisfy the constraint (seting $\ell = 1$ from now on),

$$x^+x^- = \frac{1}{2}(1 + r^2), \tag{3.19}$$

$(r^2 = x^ix^i)$. Using the definition of H_o and N in terms of $SO(2, d+1)$ generators, we get

$$
\begin{aligned}
X^i &= x^i, \\
\omega Z^- &= -x^+ \sin \omega t - \xi \omega x^- \cos \omega t, \\
X^+ &= x^+ \cos \omega t - \xi \omega x^- \sin \omega t, \\
Z^+ &= -\omega x^- \sin \omega t, \\
\omega X^- &= \omega x^- \cos \omega t,
\end{aligned}
\tag{3.20}
$$

for the relation between the embedding coordinates and the coordinates $(t, \xi, x^\pm, x^i)$ adapted to the NRCFT. The coordinates $(t, \xi, x^\pm, x^i)$ with the above ranges cover the AdS hyperboloid once. In order to remove closed timelike curves, we unwrap the time coordinate t by letting it go over $-\infty < t < \infty$. As in section 3.1, the discreteness of the spectrum of N suggests that the direction along the coordinate ξ be compact. This is achieved by identifying points $\xi \sim \xi + \lambda$, for some constant λ. The ambient metric eq. (3.17) in these coordinates is

$$ds^2 = -2\omega^2 x^-(x^- d\xi + x^+ dt)dt - 2dx^+dx^- + dx^idx^i, \tag{3.21}$$

and the AdS metric is the induced metric on points that satisfy the constraint eq. (3.19). On surfaces of constant (t, ξ), the metric is $ds^2 = -2dx^+dx^- + dx^idx^i$, and thus the coordinates $(x^\pm, x^i)$ subject to eq. (3.19) span a copy of d-dimensional hyperbolic space $\mathbf{H}^d$. Eqs. (3.18), (3.21) describe a locally AdS spacetime with global topology $\mathbf{S}^1{}_\xi \times \mathbf{R}^{d+1}$.

To study the boundary (where the NRCFT lives), recall that in AdS this consists of the re-scaled points $(X^\pm, Z^\pm, X^i) \rightarrow s(X^\pm, Z^\pm, X^i)$, $s \rightarrow \infty$ of the quadric eq. (3.16). These points satisfy

$$0 = 2X^+X^- + 2Z^+Z^- + X^iX^i, \tag{3.22}$$

with the identification $(X^\pm, Z^\pm, X^i) \sim s(X^\pm, Z^\pm, X^i)$ under (now finite) scaling. On the coordinates $(t, \xi, x^\pm, x^i)$, this re-scaling gives

$$(t, \xi, x^\pm, x^i) \rightarrow (t, \xi, sx^\pm, sx^i). \tag{3.23}$$

Thus the boundary consists of points spanned by (t, ξ) as well as $(x^\pm, x^i)$, with the equivalence $(x^\pm, x^i) \sim s(x^\pm, x^i)$ under re-scalings, and subject to the constraint

$$2x^+ x^- = r^2. \tag{3.24}$$

The set of points satisfying the relation consists of an $\mathbf{S}^{d-1}$, which is the boundary at infinity of hyperbolic space $\mathbf{H}^d$ as described by eq. (3.19). Thus the boundary of the spacetime defined by eq. (3.18) is conformal to the space $\mathbf{R}_t \times \mathbf{S}^1{}_\xi \times \mathbf{S}^{d-1}$.

For example, using the equivalence under scaling, one can set, e.g., $x^- = 1/2$, and thus $x^+ = r^2$ by eq. (3.24). The metric on the boundary becomes (up to an overall factor of $s^2 \rightarrow \infty$)

$$ds_{bd}^2 = -\omega^2 r^2 dt^2 - \frac{1}{2}\omega^2 dt d\xi + dx^i dx^i. \tag{3.25}$$

Up to normalization of ξ, this is exactly the metric in eq. (2.21). From the discussion in section 2.2, this indicates that the boundary NRCFT lives in a background harmonic confining trap. Because eq. (2.21) is conformally flat, this result is consistent with the boundary conformal structure of AdS. Alternatively, we can choose to set $x^+ = 1/2$ on the boundary by rescaling. In that case, $x^- = r^2$ and the boundary metric becomes

$$ds_{bd}^2 = -\omega^2 r^2 dt^2 - 2\omega^2 r^4 d\xi dt + dx^i dx^i. \tag{3.26}$$

This metric is conformally equivalent to the metric in the patch $x^- = 1/2$, as can be seen by performing an inversion $x^i \rightarrow x^i/2r^2$ and pulling out a factor of r^4. The full boundary is the union of the two patches $x^\pm = 1/2$.

As it stands $x^\pm$ are not independent variables. A convenient set of of coordinates on the surface eq. (3.19) is x^i together with $z = x^+/x^-$. In terms of $x^\mu = (t, \xi, z, x^i)$ the metric is now,

$$ds^2 = -\omega^2(1 + r^2)\left(dt^2 + \frac{dt d\xi}{z}\right) + \frac{1}{4}(1 + r^2)\frac{dz^2}{z^2} + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{d-2}^2, \tag{3.27}$$

with $0 < z < \infty$. The boundary of the spacetime can be reached by taking $r \rightarrow \infty$. To put the resulting metric in the form of the oscillator metric eq. (2.21), one has to re-define $z = \rho^2$ and make the conformal re-scaling $ds^2 \rightarrow \rho^2 ds^2$. Hypersurfaces Σ_t of constant t are lightlike, with normal vector $k = \partial_\xi$. The generators of Σ_t are the null geodesics with tangent vector k, one passing through every fixed value of (z, x^i).

3.2.2 Energy eigenstates

Having constructed the global time coordinate t, we can now check that the spectrum of H_o in the NRCFT matches the energy spectrum of $i\partial_t$ acting on AdS fields. For simplicity,

we consider the energy eigenmodes of a free massive scalar field ϕ in AdS_{d+2}. The Klein-Gordon equation in the coordinates $x^\mu = (t, \xi, z, x^i)$ is

$$\left[\frac{4}{1+r^2} \left(-\frac{z}{\omega^2} \partial_\xi \partial_t + \frac{z^2}{\omega^2} \partial_\xi^2 + z^2 \partial_z^2 \right) + \frac{1}{r^{d-2}(1+r^2)} \partial_r (1+r^2)^2 r^{d-2} \partial_r + \frac{1}{r^2} \nabla_{d-2}^2 - m^2 \right] \phi = 0, \tag{3.28}$$

plus possible terms from eq. (3.11). We will also need the norm for states quantized relative to (lightlike) hypersurfaces Σ_t of constant t ,

$$\langle \phi_1 | \phi_2 \rangle = \frac{i}{2} \int_{\Sigma_t} d\Sigma^\mu \, \phi_1^\dagger \overleftrightarrow{\partial}_\mu \phi_2 = \frac{i}{2} \int_{\Sigma_t} d\xi \, \frac{dz}{2z} \, r^{d-2} dr \, d\Omega_{d-2} \, \phi_1^\dagger \overleftrightarrow{\partial}_\xi \phi_2. \tag{3.29}$$

In the second equality, we have used the definition $d\Sigma^\mu = k^\mu \sqrt{h} d^{d+1}x$, with h the determinant of the metric on a d-dimensional surface with $t, \xi = $ constant and $k^\mu = \delta_\xi^\mu$ is the normal vector to Σ_t (this is the definition of k^μ, $d\Sigma^\mu$ used in formulating Stokes' theorem for integrals over regions bounded by null surfaces). The inner product is t-independent as long as the flux of the current $J_\mu = i\phi_1^\dagger \overleftrightarrow{\partial}_\mu \phi_2$ through the boundary at $r \to \infty$ vanishes,

$$\mathcal{F} = r^{d+2} J_r \Big|_{r \to \infty} \to 0. \tag{3.30}$$

Plugging in the ansatz

$$\phi = e^{-iEt} e^{-iM\xi} Z(z) R(r) Y_\ell(\Omega), \tag{3.31}$$

with $Y_\ell(\Omega)$ a spherical harmonic on $\mathbf{S}^{d-2}$, eq. (3.28) factorizes into

$$\frac{1}{r^{d-2}(1+r^2)} \frac{d}{dr} (1+r^2)^2 r^{d-2} \frac{d}{dr} R + \left(\frac{L^2}{r^2} + m^2 - \frac{\lambda}{1+r^2} \right) R = 0, \tag{3.32}$$

and

$$\frac{d^2 Z}{dz^2} + \left(-\frac{M^2}{\omega^2} + \frac{ME}{\omega^2 z} - \frac{\lambda}{4z^2} \right) Z = 0. \tag{3.33}$$

The effect of terms such as in eq. (3.11) is to shift the bulk mass m by some polynomial in M. Here λ is a constant yet to be determined, and $L^2 = \ell(\ell + d - 3)$ is minus the eigenvalue of ∇_{d-2}^2 acting on $Y_\ell(\Omega)$. The solution of eq. (3.32) that is regular at $r = 0$ is the hypergeometric function

$$R(r) = r^\ell (1+r^2)^{-\Delta/2 - \ell/2} {}_2F_1 \left(a, b; c; \frac{r^2}{1+r^2} \right), \tag{3.34}$$

with

$$\begin{aligned} a &= \frac{1}{2}(\ell + \Delta - 1) + \frac{1}{2}\sqrt{1+\lambda}, \\ b &= \frac{1}{2}(\ell + \Delta - 1) - \frac{1}{2}\sqrt{1+\lambda}, \\ c &= \ell + \frac{d-1}{2}. \end{aligned} \tag{3.35}$$

The parameter Δ is given by one of the roots in eq. (3.10). For eq. (3.33), the solution that is regular at $z \to 0$ can be written in terms of the confluent hypergeometric function

$$Z(z) = z^{\mu+1/2} e^{-\omega z/4M} {}_1F_1\left(\mu - \nu + \frac{1}{2}; 2\mu + 1; \frac{\omega z}{2M}\right) \tag{3.36}$$

with $\mu = \frac{1}{2}\sqrt{1+\lambda}$, $\nu = E/2\omega$.

Near the boundary of spacetime, the radial function has the behavior

$$R(r \to \infty) \sim \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(b)\Gamma(a)} r^{\Delta-d-1} + \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} r^{-\Delta}. \tag{3.37}$$

Normalizability with respect to eq. (3.29) is only possible if $-b$ is a non-negative integer, in which case the term that scales like $r^{\Delta-d-1}$ vanishes. The remaining term has finite norm as long as

$$\Delta \geq \frac{d-1}{2}, \tag{3.38}$$

and thus we recover the unitarity bound on NRCFT scaling dimensions from the gravity description.[5] This choice of parameters also implies that the flux eq. (3.30) of the Klein-Gordon current vanishes at the boundary, ensuring that time evolution along Σ_t hypersurfaces is unitary. In addition, finiteness of the z integral in eq. (3.29) requires that $\mu - \nu + 1/2$ in eq. (3.36) must be a negative integer or zero. We can now eliminate λ to obtain a relation between E and Δ,

$$\frac{E}{2\omega} = \frac{1}{2}(\Delta + \ell - 1) + n, \quad (n = 0, 1, 2, \cdots). \tag{3.39}$$

Up to a zero-point constant, this result reproduces all the features of the NR state-operator correspondence developed in [9]. A primary operator $\mathcal{O}(0)$ of the boundary NR-CFT is dual to the $\ell = 0$ mode of the field ϕ and corresponds to the lowest level in an $SL(2,\mathbf{R})$ ladder of oscillator states with spacing $\Delta E = 2\omega$. Higher multipoles of ϕ correspond to the states $\mathcal{O}_{i_1\cdots i_\ell} = \partial_{i_1}\cdots\partial_{i_\ell}\mathcal{O}(0) - \text{traces}$, of scaling dimension $\Delta + \ell$. Eq. (3.39) confirms that the duality between NRCFT operators and AdS bulk fields proposed in section 3.1 correctly reproduces the global aspects of Schrodinger invariant field theory.

4 Conclusion

In this paper we have formulated a correspondence between (semiclassical) gravity in $d+2$ dimensions and non-relativistic conformal field theory in $d-1$ spatial dimensions. In this approach, the $Sch(d-1)$ symmetry of the NRCFT manifests itself on the gravity side as the residual symmetry of AdS that remains after projecting onto eigenmodes along a fixed null direction. It becomes possible to set up the correspondence directly in (locally) AdS

---

[5]The bound $\Delta \geq \frac{d-1}{2}$ on scalar NRCFT primary operators can be obtained from the highest weight representation theory of the $SL(2,\mathbf{R})$ subgroup of the Schrodinger algebra discussed in section 2.2. It arises by demanding positivity of norm to the "level-2" subspace generated by states $L_+|\mathcal{O}\rangle$, $(\mathbf{P}+i\mathbf{K})|\mathcal{O}\rangle$ of dimension $\Delta + 2$.

spacetimes, avoiding the need to introduce non-vacuum sources of energy-momentum (i.e., sources in addition to cosmological constant).

Here we have set up the basic dictionary (similar to the usual AdS/CFT dictionary) and tested the consistency of the approach by showing that it reproduces certain properties of NRCFTs that can be derived from symmetry considerations alone (the form of two-point correlators and the state-operator correspondence). There are several theoretical issues that must still be addressed. First, in our proposal (and also in the proposal of [14, 15]) for constructing Schrodinger invariant spacetimes, it seems necessary to compactify the null coordinate ξ dual to the orbits of the particle number generator N. However, lightlike compactifications are problematic both in field theory and gravity. See [31] for a discussion. Second, the boundary limit of the spacetime geometry considered in this paper does not satisfy the criteria discussed in ref. [32] that are necessary for causality in the boundary theory. However, if one takes the point of view that the gravity duals considered in this paper should be regarded as phenomenological models of strongly interacting approximate NRCFTs (in the spirit of "AdS/QCD" models of strong interaction physics), then the necessity of satisfying boundary causality seems not to be crucial. Indeed, in the theory of fermions near unitarity that originally motivated this paper, the Lagrangian contains non-local short range interactions, and causality requirements such as those of ref. [32] do not strictly hold.

Although we have only explicitly considered a toy gravity theory containing a single dynamical scalar, it is straightforward to extend our analysis to more complicated gravity models. In particular, to reproduce the operator spectrum of realistic NRCFTs it necessary to include additional fields, for example: the bulk graviton $h_{\mu\nu}$ (related to the various components of $T_{\mu\nu}$ in the boundary theory), a Kaluza-Klein gauge field resulting from lightlike compactification (dual to the particle number current operator), bulk gauge fields related to the global symmetries of the boundary CFT.

It remains to be seen whether the AdS/NRCFT proposal presented here can be used to make predictions for the dynamics of strongly coupled NRCFTs probed in the laboratory. As a first step in that direction, it is necessary to establish a version of the correspondence for NRCFTs at finite temperature and density. It seems clear that this can be done by studying gravity duals propagating in asymptotically AdS (charged) black hole backgrounds, compactified on a suitable chosen lightlike direction in order to generate the appropriate non-relativistic symmetries. In that case it should be possible to apply the existing AdS/CFT technology for e.g., computing finite temperature/density response functions and hydrodynamic transport properties [33–36].

Acknowledgments

References

[1] E. Braaten and H.W. Hammer, *Universality in few-body systems with large scattering length*, *Phys. Rept.* **428** (2006) 259 [`cond-mat/0410417`] [SPIRES].

[2] For a theoretical review, as well as references to the experimental literature, see S. Giorgini, L.P. Pitaevskii and S. Stringari, *Theory of ultracold Fermi gases*, `arXiv:0706.3360`.

[3] C.R. Hagen, *Scale and conformal transformations in galilean-covariant field theory*, *Phys. Rev.* **D 5** (1972) 377 [SPIRES].

[4] U. Niederer, *The maximal kinematical invariance group of the free Schrödinger equation*, *Helv. Phys. Acta* **45** (1972) 802 [SPIRES].

[5] T. Mehen, I.W. Stewart and M.B. Wise, *Conformal invariance for non-relativistic field theory*, *Phys. Lett.* **B 474** (2000) 145 [`hep-th/9910025`] [SPIRES].

[6] D.B. Kaplan, M.J. Savage and M.B. Wise, *A new expansion for nucleon nucleon interactions*, *Phys. Lett.* **B 424** (1998) 390 [`nucl-th/9801034`] [SPIRES]; *Two-nucleon systems from effective field theory*, *Nucl. Phys.* **B 534** (1998) 329 [`nucl-th/9802075`] [SPIRES].

[7] R. Jackiw and S.-Y. Pi, *Classical and quantal nonrelativistic Chern-Simons theory*, *Phys. Rev.* **D 42** (1990) 3500 [*Erratum ibid.* **48** (1993) 3929] [SPIRES].

[8] M. Henkel, *Schrödinger invariance in strongly anisotropic critical systems*, *J. Stat. Phys.* **75** (1994) 1023 [`hep-th/9310081`] [SPIRES].

[9] Y. Nishida and D.T. Son, *Nonrelativistic conformal field theories*, *Phys. Rev.* **D 76** (2007) 086004 [`arXiv:0706.3746`] [SPIRES].

[10] Y. Castin, *Basic theory tools for degenerate Fermi gases*, `cond-mat/0612613`.

[11] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [`hep-th/9711200`] [SPIRES].

[12] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [`hep-th/9802109`] [SPIRES].

[13] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [`hep-th/9802150`] [SPIRES].

[14] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schrödinger symmetry*, *Phys. Rev.* **D 78** (2008) 046003 [`arXiv:0804.3972`] [SPIRES].

[15] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601 [`arXiv:0804.4053`] [SPIRES].

[16] G.T. Horowitz and H. Ooguri, *Spectrum of large-N gauge theory from supergravity*, *Phys. Rev. Lett.* **80** (1998) 4116 [`hep-th/9802116`] [SPIRES].

[17] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [`hep-th/9803131`] [SPIRES].

[18] J.P. Gauntlett, J. Gomis and P.K. Townsend, *Supersymmetry and the physical phase space formulation of spinning particles*, *Phys. Lett.* **B 248** (1990) 288 [SPIRES]; C. Duval and P.A. Horvathy, *On Schrödinger superalgebras*, *J. Math. Phys.* **35** (1994) 2516 [`hep-th/0508079`] [SPIRES];

M. Henkel and J. Unterberger, *Supersymmetric extensions of Schrödinger-invariance*, *Nucl. Phys.* **B 746** (2006) 155 [`math-ph/0512024`] [SPIRES];
M. Sakaguchi and K. Yoshida, *Super Schrödinger in super conformal*, `arXiv:0805.2661` [SPIRES].

[19] G. Burdet, J. Patera, M. Perrin and P. Winternitz, *The optical group and its subgroups*, *J. Math. Phys.* **19** (1978) 1758 [SPIRES].

[20] J. Gomis and J.M. Pons, *Poincaré transformations and Galilei transformations*, *Phys. Lett.* **A 66** (1978) 463 [SPIRES];
J. Gomis, A. Poch and J.M. Pons, *Poincaré wave equations as Fourier transforms of Galilei wave equations*, *J. Math. Phys.* **21** (1980) 2682 [SPIRES].

[21] C. Duval, G. Burdet, H.P. Kunzle and M. Perrin, *Bargmann structures and Newton-Cartan theory*, *Phys. Rev.* **D 31** (1985) 1841 [SPIRES].

[22] M. Henkel and J. Unterberger, *Schrödinger invariance and space-time symmetries*, *Nucl. Phys.* **B 660** (2003) 407 [`hep-th/0302187`] [SPIRES].

[23] T. Mehen, *On non-relativistic conformal field theory and trapped atoms: virial theorems and the state-operator correspondence in three dimensions*, `arXiv:0712.0867` [SPIRES].

[24] C. Duval, P.A. Horvathy and L. Palla, *Conformal symmetry of the coupled Chern-Simons and gauged nonlinear Schrödinger equations*, *Phys. Lett.* **B 325** (1994) 39 [`hep-th/9401065`] [SPIRES].

[25] C. Duval, P.A. Horvathy and L. Palla, *Conformal properties of Chern-Simons vortices in external fields*, *Phys. Rev.* **D 50** (1994) 6658 [`hep-ph/9405229`] [SPIRES].

[26] M. Lüscher and G. Mack, *Global conformal invariance in quantum field theory*, *Commun. Math. Phys.* **41** (1975) 203 [SPIRES].

[27] R. Jackiw, *Dynamical symmetry of the magnetic monopole*, *Ann. Phys.* **129** (1980) 183 [SPIRES]; *Dynamical symmetry of the magnetic vortex*, *Ann. Phys.* **201** (1990) 83 [SPIRES].

[28] F. Werner and Y. Castin, *Phys. Rev.* **A 74** (2006) 053604 [`cond-mat/0607821`] [SPIRES].

[29] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [`hep-th/9905104`] [SPIRES].

[30] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Ann. Phys.* **144** (1982) 249 [SPIRES].

[31] S. Hellerman and J. Polchinski, *Compactification in the lightlike limit*, *Phys. Rev.* **D 59** (1999) 125002 [`hep-th/9711037`] [SPIRES].

[32] V.E. Hubeny, M. Rangamani and S.F. Ross, *Causal structures and holography*, *JHEP* **07** (2005) 037 [`hep-th/0504034`] [SPIRES].

[33] G. Policastro, D.T. Son and A.O. Starinets, *The shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **87** (2001) 081601 [`hep-th/0104066`] [SPIRES].

[34] D. Birmingham, I. Sachs and S.N. Solodukhin, *Conformal field theory interpretation of black hole quasi-normal modes*, *Phys. Rev. Lett.* **88** (2002) 151301 [`hep-th/0112055`] [SPIRES].

[35] D.T. Son and A.O. Starinets, *Minkowski-space correlators in AdS/CFT correspondence:*

recipe and applications, JHEP **09** (2002) 042 [`hep-th/0205051`] [SPIRES];
C.P. Herzog and D.T. Son, *Schwinger-Keldysh propagators from AdS/CFT correspondence,*
JHEP **03** (2003) 046 [`hep-th/0212072`] [SPIRES].

[36] G. Policastro, D.T. Son and A.O. Starinets, *From AdS/CFT correspondence to*
hydrodynamics, JHEP **09** (2002) 043 [`hep-th/0205052`] [SPIRES]; *From AdS/CFT*
correspondence to hydrodynamics. II: sound waves, JHEP **12** (2002) 054 [`hep-th/0210220`]
[SPIRES].